Calculation of α_{cr} for unbraced frames

In this article, Dr. Yigit Ozcelik of the Steel Construction Institute (SCI) presents a simple yet efficient hand method to estimate the global stability parameter, α_{cr}, for unbraced frames.

Introduction

In accordance with BS EN 1993-1-1[1] Clause 5.2.1, either a first-order or second-order analysis can be used to determine internal forces and moments, providing the criteria for the chosen method are satisfied. For first-order elastic analysis, the criterion requires that the factor, α_{cr}, be greater than or equal to 10. This factor represents the multiplier by which the design loading would need to be increased to cause elastic instability in a global mode (see Equation (1)). If α_{cr} falls between 3 and 10, second-order effects may be considered using an approximate second-order analysis. However, for structures where α_{cr} is less than 3, a rigorous second-order analysis is required.

$$\alpha_{cr} = \frac{F_{cr}}{F_{ed}} \quad (1)$$

where F_{Ed} is the design loading on the structure
F_{cr} is the elastic critical buckling load for global instability mode based on initial elastic stiffness

To calculate F_{cr} (and α_{cr}) precisely, a linear buckling analysis is normally needed; however, BS EN 1993-1-1[1] introduces an approximate method to estimate α_{cr} on a storey-by-storey basis within a building:

$$\alpha_{cr} = \left(\frac{H_{Ed}}{V_{Ed}}\right)\left(\frac{h}{\delta_{H,Ed}}\right) \quad (2)$$

where H_{Ed} is the total design horizontal load transferred by the storey
V_{Ed} is the total design vertical load on the frame transferred by the storey
$\delta_{H,Ed}$ is the horizontal displacement at the top of storey relative to the bottom of the storey when the frame is loaded with horizontal loads
h is the storey height

Similar to the linear buckling analysis, the approximate method also requires the use of structural analysis software to calculate horizontal displacements at storey levels. While such software is indispensable in modern engineering practice, a lack of understanding of its underlying assumptions can lead to erroneous results. Therefore, hand calculations remain a valuable practice to verify software output. In this article, a simple hand method based on first principles is introduced to calculate α_{cr} for unbraced frames.

Background on elastic critical buckling load

The elastic critical buckling load, N_{cr}, is defined as the compressive load at which an elastic column will suddenly bend and buckle.

$$N_{cr} = \frac{\pi^2 EI}{L^2} \quad (3)$$

where E is the modulus of elasticity
I is the second moment of area
L is the length

Equation (3) was derived by Leonhard Euler in 1744, writing the equations of equilibrium of a pin-ended column in the deformed configuration, using the Euler-Bernoulli beam theory, which describes the relationship between deflection and applied load.

The effective length factor, K, commonly referred to as the K-factor, is a multiplier that enables the calculation of an artificial column length that allows the use of Euler's equation to evaluate the elastic critical buckling load of a column with relatively general support conditions (Figure 1). This leads to the general form of Euler's formula:

$$N_{cr} = \frac{\pi^2 EI}{(KL)^2} \quad (4)$$

K-factors were determined for idealised end conditions such as pinned–pinned, fixed–fixed, pinned–fixed, and fixed–free, and are widely available in literature. However, these ideal cases have limited practical value in real-world applications, where support conditions and stiffness distributions are more complex.

For braced frames, a conservative design approach typically assumes K=1 for most situations. In practice, K<1.0 can be achieved in systems with very high lateral stiffness, but the use of unity is often recommended for simplicity and safety.

In contrast, determining appropriate K-factors for unbraced frames is more complex. In such cases, the K-factor can theoretically vary from 1.0 up to infinity, depending on the degree of rotational restraint provided by the surrounding frame. As a result, no universally applicable approach exists.

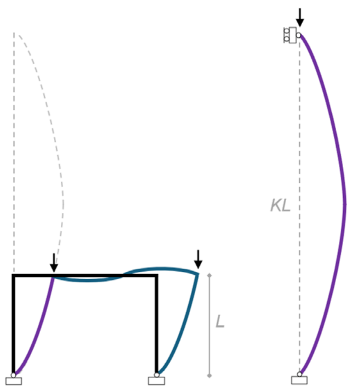

Figure 1: Column length (L) vs column effective buckling length (KL)

One approach to determining K-factors is the alignment chart that is a well-established graphical tool widely used by engineers. There are two nomographs available —one for braced frames and one for unbraced frames. The nomograph applicable to unbraced frames is shown in Figure 2.

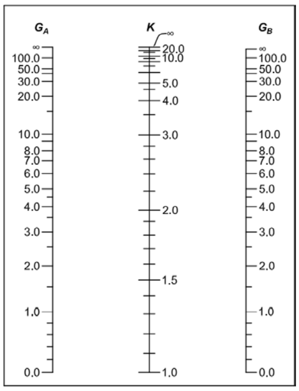

Figure 2: Alignment chart – unbraced frames

To use the nomograph, the degree of restraint at both ends of a column — denoted as G—must first be calculated using Equation (5):

$$G = \frac{\Sigma(I_c/L_c)}{\Sigma I_b/L_b} \quad (5)$$

▸26

where $\sum(I_c/L_c)$ is the sum of the ratio of the second moment of area to the length of all columns connected to the joint

$\sum(I_b/L_b)$ is the sum of the same ratio for all beams connected to the joint

As an alternative to the graphical nomograph, the following closed-formed equation may be used to calculate K-factors for unbraced frames:

$$\frac{G_A G_B \left(\frac{\pi}{K}\right)^2 - 36}{6(G_A G_B)} - \frac{\left(\frac{\pi}{K}\right)}{\tan\left(\frac{\pi}{K}\right)} = 0 \qquad (6)$$

where G_A is the degree of restraint at one end of the column (see Equation (5))

G_B is the degree of restraint at the other end of the column (see Equation (5))

It is important to recognise that the alignment chart is derived from an elastic sidesway stability analysis of a highly idealised frame under simplified loading conditions. These assumptions, along with the modifications to the alignment chart, for unbraced frames will be explored in a forthcoming article by SCI.

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Worked example 1

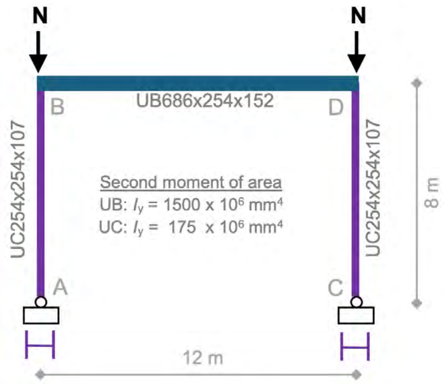

Figure 3: Worked example 1

In this example, an unbraced frame subjected to two equal vertical point loads acting at beam-column joints was evaluated to determine the critical vertical load, N, that leads to instability of the frame.

The degree of restraint for Column AB at Point B, G_B, is:

$$G_B = \left(\frac{\Sigma I_c/L_c}{\Sigma I_b/L_b}\right)_B = \frac{175 \times 10^6\text{mm}^4 / 8\text{m}}{1500 \times 10^6\text{mm}^4 / 12\text{m}} = 0.175 \qquad (7)$$

where I_c is the second moment of area of Column AB

L_c is the length of Column AB

I_b is the second moment of area of Beam BD

L_b is the length of Beam BD

Due to the pinned base, the degree of restraint for Column AB at the column base (Point A), G_A, is infinity.

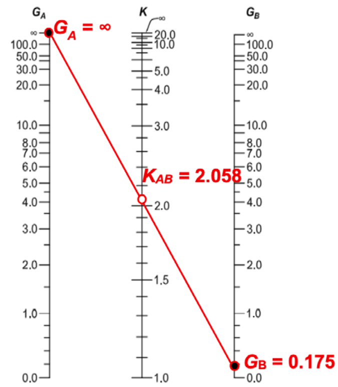

Figure4: Effective length factor for Column AB

Entering G_A and G_B into the alignment chart, the effective length factor for Column AB, K_{AB}, is 2.058.

Using Equation (4), the elastic critical buckling load for Column AB, $N_{cr,AB}$ is:

$$N_{cr,AB} = \frac{\pi^2 EI_c}{(K_{AB}L_c)^2} = \frac{\pi^2\ (210\text{kN/mm}^2)\ (175 \times 10^6\text{mm}^4)}{(2.058 \times 8000\text{mm})^2} = 1338\text{kN} \qquad (8)$$

Accordingly, N=1338 kN.

The unbraced frame was also analysed using MASTAN2 , a free structural analysis program capable of performing linear buckling analysis. The results of the analysis yielded a critical vertical load of N=1335 kN, which suggests the simple hand calculation provided an accurate prediction of the critical load, closely matching the numerical results.

Worked example 2

In this example, the unbraced frame considered in the Worked Example 1 was modified to the extent that the load distribution among the columns is different, while the total load acting on the frame remains the same.

As the alignment chart used to determine the K-factor does not account for individual column loads, the K-factor remains unchanged. Consequently, $N_{cr,AB}$ = 1338 kN also remains unchanged.

Given that the elastic critical buckling load for Column CD was calculated using the alignment chart, $N_{cr,CD}$, is equal to $N_{cr,AB}$, one might argue that Column CD would buckle first as it is subjected to a larger vertical load than

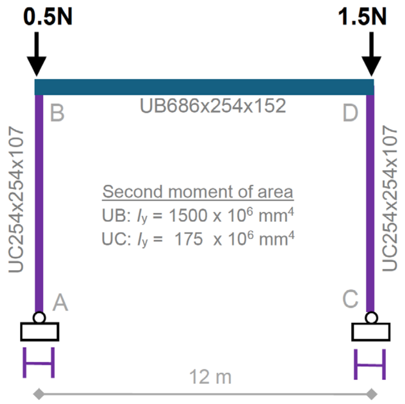

Figure 4: Worked example 2

Column AB. This would suggest that N should be lower than in the Worked Example 1. However, the linear buckling analysis of the frame with modified loads yielded the same critical vertical load: N=1335 kN.

This outcome can be explained by the fact that, when Column CD is onset of buckling, Column AB—being subjected to a smaller vertical load—still has reserve load-carrying capacity. This reserve capacity contributes to the overall stability of the frame by effectively *helping* Column CD to resist a larger load than its $N_{cr,CD}$ value. This phenomenon is known as the *ΣP Concept*[3] , which describes how, in sway buckling, *some* columns *help others* while *others* reduce the capacity of *some*, until all columns buckle together in a global sway mode. Therefore, it is not suitable to assess the sidesway stability of columns in isolation; rather, the stability of the entire storey in the sway mode must be evaluated.

According to the results of the linear buckling analysis, the critical vertical load of the frame (or storey) is 2N=2670 kN. Using this value, the effective length factors of Column AB and Column CD, (K_{AB} and K_{CD}, respectively) were back-calculated:

$$K_{AB} = \sqrt{\frac{\pi^2 EI_c}{0.5NL_c^2}} = \sqrt{\frac{\pi^2\,(210\text{kN/mm}^2)(175\times10^6\text{mm}^4)}{(0.5\times1335\text{kN})(8000\text{mm})^2}} = 2.914 \quad (9)$$

$$K_{CD} = \sqrt{\frac{\pi^2 EI_c}{1.5NL_c^2}} = \sqrt{\frac{\pi^2\,(210\text{kN/mm}^2)(175\times10^6\text{mm}^4)}{(1.5\times1335\text{kN})(8000\text{mm})^2}} = 1.618 \quad (10)$$

Notably, the K-factor determined from the alignment chart in the Worked Example 1 differs significantly from the values obtained in Equations (9) and (10). However, the elastic critical buckling load of the frame (or storey), $N_{cr,storey}$ —calculated as the sum of the elastic critical buckling load of each column estimated using the alignment chart according to the ΣP concept—matches the result from the linear buckling analysis. This leads to an important conclusion: the elastic buckling load of an individual column in an unbraced frame determined using an alignment chart K-factor, should be interpreted not as the maximum load that column can support, but rather as its contribution to the overall storey's buckling stiffness. Hence, $N_{cr,storey}$ can be accurately estimated using the alignment charts even if the K-factors for individual columns are not accurate:

$$N_{cr,storey} = \sum N_{cr,i} \quad (11)$$

where $N_{cr,i}$ is the elastic critical buckling load of Column i using the alignment chart K-factor

However, it is important to note that the restraint (or *help*) provided by *some* columns to *others* is limited by the elastic buckling resistance of *other* columns in the no-sway mode— that is, assuming K = 1.0. In other words, each column must be able to support its own vertical load in isolation in the no-sway mode, without relying on the help. It is worth mentioning that elastic buckling of a column (which is part of a stability system) in the no-sway mode is quite unlikely for orthodox frame configurations.

Similar to the approximate method given in BS EN 1993-1-1[1] (see Equation (2)), α_{cr} can be calculated on a storey-by-storey basis within a building:

$$\alpha_{cr} = \frac{N_{cr,storey}}{V_{Ed}} \quad (12)$$

Conclusion

In this article, a simple hand method is presented for calculating the global stability parameter, α_{cr}, of unbraced frames based on the fundamentals of the stability theory and effective length factors obtained from the alignment chart. The method allows engineers to estimate α_{cr} without relying on structural analysis software.

Through two worked examples, it was shown that the elastic critical buckling load of a storey for global instability mode—and therefore the calculated α_{cr}—remains accurate despite observing that the elastic critical buckling load of individual columns of the storey calculated using the alignment chart might be incorrect.

The method enables accurate estimation of α_{cr} and offers a valuable verification tool for engineers. ■

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