A view of Torsion – Part Two

In Part One, the two mechanisms by which a member can resist torsion were introduced. One of them, so-called 'warping', can be simply understood – not as warping but as **differential flange bending** – and, on its own, simply calculated. The other, named after **St Venant**, is simple to understand for thin-walled hollow sections, but becomes mathematically demanding for conventional sections. In Part Two, Alastair Hughes describes an escape route for non-mathematicians.

The problem

Elastic torsion resistance of 'solid' sections is not a simple matter of shear flow, as in the thin-walled hollow section considered in Part One. The shear stress will, broadly speaking, be at its highest at the periphery of the cross-section and diminish towards the middle. To take the simplest of examples, a solid round shaft could be viewed as a set of nested thin-walled hollow sections in each of which the shear flows as previously considered – but the strain, and hence the stress, would be proportional to the radius; at the centre there might as well be a small hole and the metal half way out would be 25% as effective as that at the periphery – 50% of the force per unit length at 50% of the lever arm. Without axisymmetry, the flow of the shear around the section will vary in intensity and direction according to laws of compatibility of strain. As with every elastic problem, there must be a unique solution, but in all except the simplest shapes this can only be arrived at by numerical methods. It is at this point that most of us non-mathematicians run for cover.

The membrane (or soap film) analogy

Fortunately help is at hand. A Bavarian engineer of the early 20th century, Ludwig Prandtl, as an aside from establishing the study of fluid dynamics, recognized that the equations which govern St Venant shear are identical to those which control the shape of a pressurized soap bubble that stretches across the same outline. Subject only to small deflections, the mathematical equivalence is exact.

This analogy must be one of the most potent in all engineering. Few can make sense of a set of equations but everyone can visualize the form a soap bubble will take.

Here are the instructions. Cut the cross-section outline out of a thin rigid plate which is the top of an otherwise sealed box into which air can be pumped by something like a bicycle pump after the soap bubble has been stretched across the section-shaped hole. The characteristic of the soap film is that it has constant surface tension in all directions. The bubble will inflate in proportion to the pressure. Stop pumping as soon as the form is clearly visible. The volume of air under the membrane (above base level) is proportional to the torsional moment. The shear flows along the contours. Its intensity – the shear stress – represents the slope of the membrane.

The pictures here are not of real soap bubbles but simulations, for which we are indebted to Chris Williams and Rachel Cruise of Bath University and their form-finding software. (They are not responsible for the rather vivid rendering!) The section portrayed is 406 × 178 UKB74.

For an I-section, the membrane mainly takes the form of a cylindrical barrel between the parallel sides of the flange or web. The slope is obviously greatest at the outer faces and zero at the summit at mid-thickness (where the direction of shear flow reverses). If, for example, the web is half as thick as the flange, constant surface tension demands that the slope at the web face is half what it is at the flange face. Consequently the volume under the membrane is, per unit length, **one eighth** that at the flange, where twice as much metal is working four times as hard. Only a fraction of the total torsional performance is contributed by the web.

Where the boundaries of the cross-section are not parallel, the form of the membrane becomes three-dimensional. At the flange tips, there is a tendency to span across the corners, flattening the membrane and resulting in some loss of effectiveness as the shear flow 'cuts the corner'. Conversely, the bubble domes up at the junction of flange and web, especially with a generous root radius, and typically the gain at the two junctions more than compensates for what is lost at the four flange tips.

Observe, in passing, how helpful the root radii are in keeping the bubble attached, and minimizing the stress raising effect of re-entrant corners.

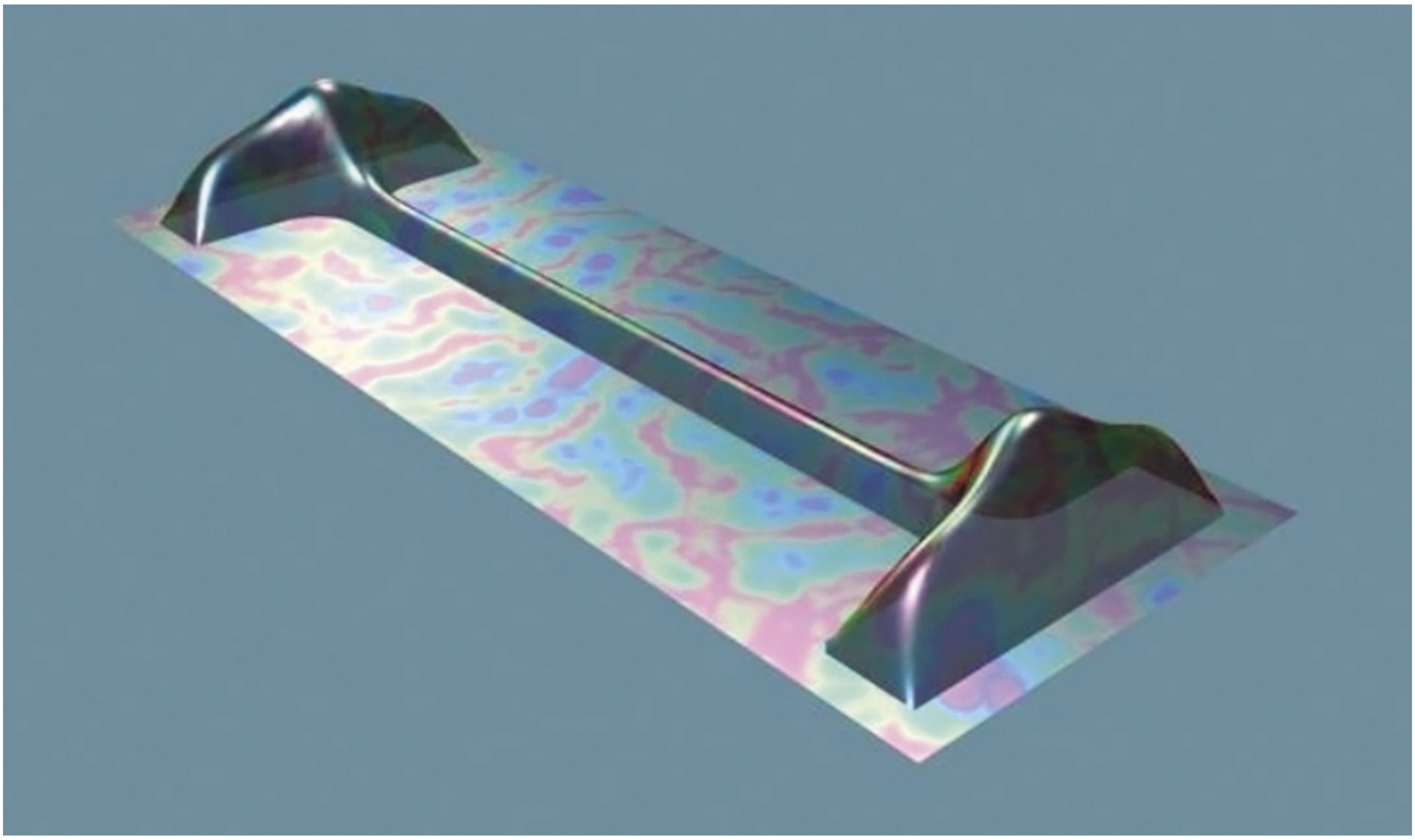
The membrane analogy has, in the not so distant past, been used for quantitative purposes. It may seem hard to believe today, but people in lab coats really did blow bubbles as described and take precise measurements of their shape in order to derive torsional properties.

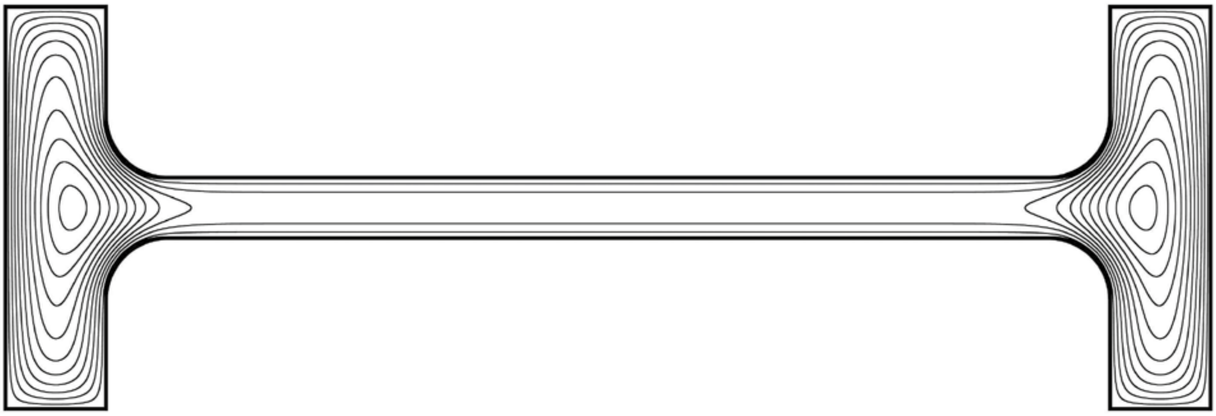
Hollow sections

The analogy extends to hollow sections, for which the loose plate cut out from within the section needs to be (i) weightless and (ii) constrained, as if on rails, to move only in the vertical direction, without rotation or translation. It rides upwards as the air is pressurized, so its whole area contributes to the volume under the bubble. In a thin walled hollow section, convexity of the membrane accounts for a very small proportion of the total resistance, and the slope, alias the shear stress, diminishes only slightly from outside to inside.

A single longitudinal cut in the hollow section would, at a stroke, keep the interior plate attached and grounded, reducing the volume of air above the baseline by a factor which depends on the thickness but could easily be as much as 100. That's another dramatic demonstration of the superiority of hollow sections. The volume due to membrane convexity is all that remains.

As a matter of interest, the analogy is good for hollow sections with more than one cell (in which the detached plates may rise to different levels) and with varying wall thickness. In a single cell hollow section, the shear stress will be lower where the wall is thicker – the reverse of the case with open sections, but obvious, in both cases, with the insight provided by the membrane analogy. Another difference is that the walls of rectangular hollow





sections with $h >> t$ are prone to torsion-induced shear buckling (in principle at least; few in the current range are slender enough to be susceptible) whereas open sections, with St Venant shear tugging the two sides of the element in opposite directions, are not.

The sandhill analogy

There is also a sandhill analogy to represent plastic torsion. The sand has a constant angle of repose (alias the yield stress in shear, $\tau_y = f_y/\sqrt{3}$) and the volume that can be heaped on the cross-section corresponds to the volume of air under the membrane (alias the torsional moment). The hollow of a hollow section will enforce a plateau, only marginally higher than its elastic counterpart. En route to full plastification, the soap bubble can be visualized as being pumped up into a roof-like shape which matches the sandhill.

A fascinating detour, but plastic torsion is of mainly academic interest – except perhaps to the designer of an expendable energy-absorbing structure.

Verification of St Venant torsional resistance

With hollow sections, it is important to recognize that one consequence of their efficiency in resisting torsion is that virtually the entire volume of metal can be mobilized close to yield, so interaction with other effects is very direct. If utilization versus torsion is 50%, 50% is left to counter regular shear and bending effects.

The design torsional moment resistance of a hollow section is the product of St Venant torsional section modulus W_t (from section property tables) and $f_y/\sqrt{3}$, though there is evidence that non-circular hollow sections cannot always achieve it under test. Parasitic warping effects may be to blame. It's comforting that serviceability nearly always governs.

With open sections, it is not helpful to talk of a torsional section modulus, and none is tabulated. For one thing, resistance is even less likely to govern. Nor is its verification a simple matter of evaluating the maximum shear stress anywhere on the surface. For example, St Venant shear stress on the surface of a web (numerically small, but additive to regular shear) gets to be calculated, whereas twice that stress just round the corner on the flange, not to mention the stress concentration at the re-entrant corner itself, might be overlooked.

It is, however, premature to discuss the verification of I-sections before coming to terms with the lengthwise interaction between the St Venant and 'warping' resistance mechanisms. That will be the subject of Part Three.