Stability and second order effects on steel structures: Part 1: fundamental behaviour

Ricardo Pimentel of the SCI introduces the topics of buckling phenomenon, second order effects and the approximate methods to allow for those effects. In part 2, the various methods will be compared to the results from a rigorous numerical analysis.

When a structure is loaded, deformation occurs, and the internal forces within the structure are modified. If at some point an increase of load (and deflection) does not modify the internal forces, the structure became unstable (only considering elastic buckling). In a perfect structure, a theoretical sudden instability exists when the applied loads reach a critical load. However, because real structures are always imperfect, the so-called sudden instability does not exist – an initial bow imperfection in a strut will increase as the applied load increases. When the applied load becomes closer to the theoretical critical value, the deformation increases rapidly. This leads to the following conclusions: (i) when loaded, a strut tends to diverge from its initial position "guided" by the initial bow imperfection; (ii) the magnitude of the initial bow imperfection will have influence in the critical load of the strut; (iii) the applied load will have impact on the deformed shape, which in turn will influence the buckling resistance of the member.

From the concepts explained above, the assessment of instability problems must consider the effects of the deformations due to the applied loads. Even for the theoretically perfect structures, the prediction of the load that leads to sudden instability requires the assumption of a deformed shape of the system. To address the problem, taking the frame in Figure 1 as example, two types of effects are important:

(i) P-δ effects, which are related to deformations within the length of members, and

(ii) P-Δ effects, which are related to movement of nodes.

The impact of the P-δ and P-Δ effects is to change the forces and deflections within the structure. These are second order effects, not accounted for in a usual first order analysis. Second order effects may be accounted for by a geometric non-linear analysis or by approximate modifications of a first order analysis.

A second order analysis can be done through a series of first order analyses, applying the load in small increments, but for each increment, the deformed shape of the structure is considered.

For an idealized "perfect" pin-ended strut (Figure 2), the theoretical critical load that leads to a sudden instability of the system can be obtained by solving a second order differential equation[1]. In the process, the displacement "y" along "z" is established using a sinusoidal function, which later leads to the following definition:

$$P = \frac{n^2\pi^2 EI}{l^2} \quad \text{where } n=1,2,3\ldots$$

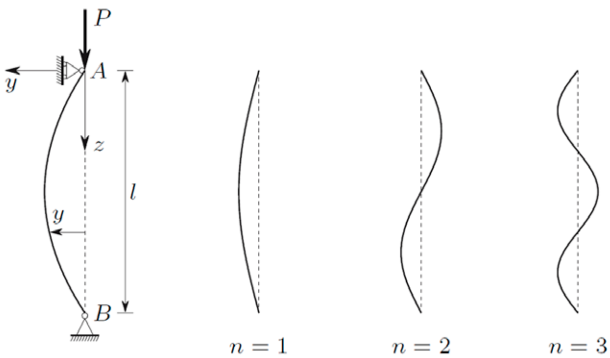

Figure 2 – Buckling modes for a pin-ended strut[2].

The load P is the Euler buckling load. It is clear that there are many possible values for P with different value of "n" leading to different buckling mode shapes. These modes are usually called

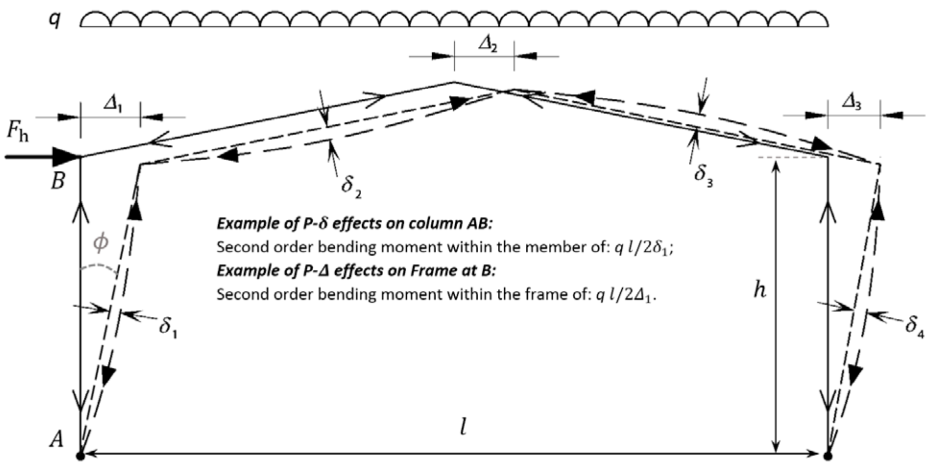

Figure 1 – Local (δ) and global (Δ) displacements which produce second order effects P-δ and P-Δ.

eigenvalues. The minimum value of P (n=1), represents the critical load of the strut (P_{cr}), which means that the first eigenvalue of the system will represent the critical buckling mode shape.

The governing equation can be re-arranged for different boundary conditions as presented in Figure 3. For some configurations (such as "a", "b" or "c"), with geometric/symmetric considerations a solution is possible without solving the differential equation. For example "a", it is clear that the critical configuration has the same shape of a pin-ended member with an equivalent length of $2l$. The corresponding critical load for case "a" is presented in the expression below ($P_{cr,a}$). The term l_{eff} is the so-called effective length, which may be defined as the length that a pin-ended strut with the same cross-section that has the same Euler load as the member under consideration.

$$P_{cr,a} = \frac{n^2\pi^2 EI}{2l^2} \text{ or } = \frac{n^2\pi^2 EI}{l_{eff}^{\ 2}} \text{ ,therefore } l_{eff}=2l$$

The behaviour presented above represents a "perfect" strut. However, imperfections will always exist, creating additional flexure in the element. This will limit the resistance to loads lower than the Euler load (line HJ in Figure 4). The residual stresses due to manufacture processes will also contribute to a lower resistance. Eurocode 3 deals with initial imperfections by specifying an equivalent bow imperfection which allows for all these effects. The behaviour of a real strut can be represented by line OCFD in Figure 4, where it is clear that the maximum axial resistance is between the elastic (Point C) and the plastic resistance of the cross section (Point G). As the resistance of Point F is difficult to determine, the calculated resistance is conservatively taken as Point C. According to clause NA.2.11 of the UK NA to EN 1993-1-1[3], to obtain the initial bow imperfection, the designer should complete a back-calculation using the buckling design procedure according to EN 1993-1-1[4] section 6.3. For the reasons explained, the elastic section modulus should be used in the process.

Figure 5 shows the Euler buckling curve (presented as stresses) which is an upper limit to the resistance. AB represents the plateau where according to theory, there is no buckling. At slenderness λ, Point G would represent the theoretical resistance, but this is reduced to Point H, due to the effect of local imperfections.

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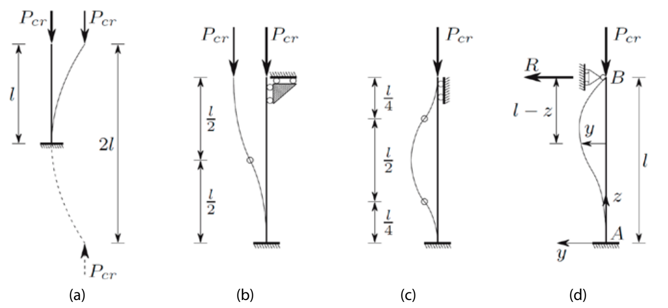
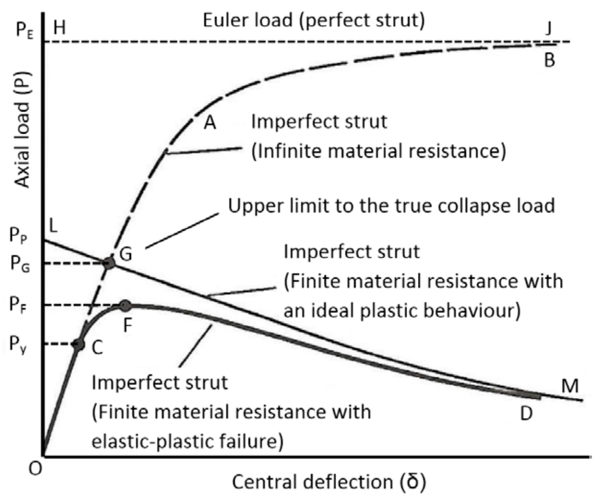


$l_{eff,a} = 2l$
$l_{eff,b} = l$
$l_{eff,c} = 0.5l$
$l_{eff,d} \approx 0.7l$

Figure 3 – Effective length for struts with different boundary conditions[2].



Notes: for an imperfect strut with finite material resistance (curve OCFD), after reaching yield (Point C), there is a clear decrease of stiffness due to plasticity, making the behaviour diverge from the elastic response (line OCG).

P – Axial Load;
P_E – Euler Load;
P_y – Load to elastic resistance;
P_F – Load in failure with elastic-plastic behaviour;
P_P – Load to ideal plastic resistance (squash load);
P_G – Load in failure with a perfect plastic hinge;
σ_y – Yield strength of the material.

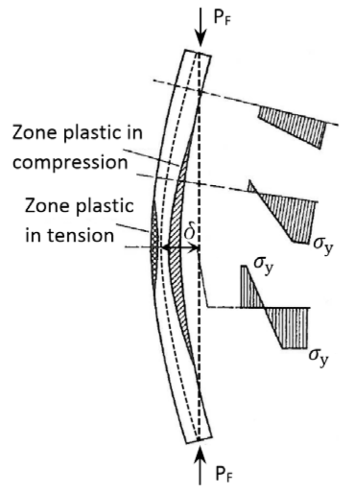
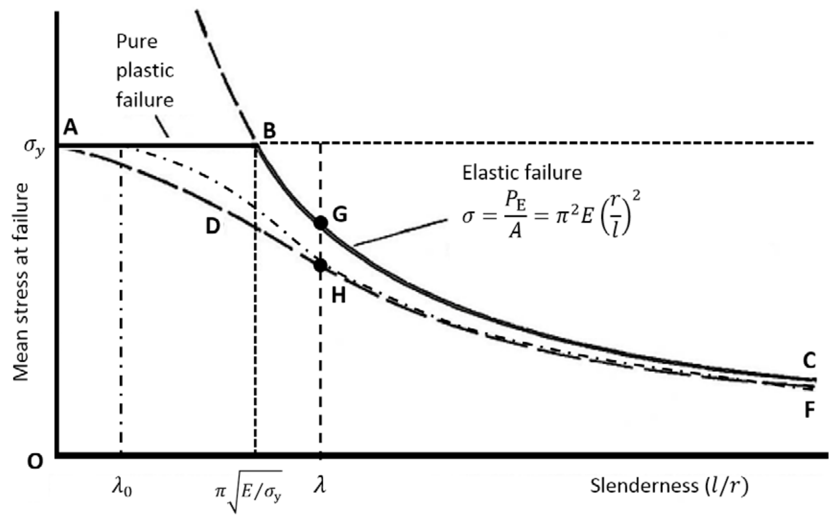


Figure 4 – Response of a strut under axial load[5,6]



P_E – Euler Load;
σ_y – Yield strength of the material.
σ – Allowable stress;
l – Strut length;
r – Radius of gyration;
λ – Slenderness;
E – Young modulus;
A – Section Area.

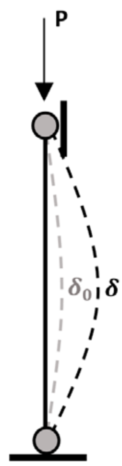


Figure 5 – Response of a real strut under axial load[5]

▶23 The Eurocode introduces an initial plateau (limited by λ_0 in Figure 5) for the design of imperfect struts. According to clause 6.3.1.3 of EN 1993-1-1, the plateau is determined by $\bar{\lambda} = 0.20$, where $\bar{\lambda} = \sqrt{A\sigma_y/P_{cr}}$ (the Eurocode terms are $\bar{\lambda} = \sqrt{Af_y/N_{cr}}$). This plateau makes an allowance for strain hardening in short columns[6]. For values above the specified slenderness for the plateau, second-order *P-δ* effects are always relevant for members.

The differential equation for the "perfect" struts in Figure 2 can be adapted to consider an initial bow imperfection. If the formulation for a "perfect" problem is rather complex, including an initial imperfection would certainly be more so. However, to demonstrate the concept of the effects of an initial bow imperfection, a simplified model can be adopted, where the system from Figure 2 is replaced by an idealized problem having a joint with a spring stiffness as shown in Figure 6 [2,6].

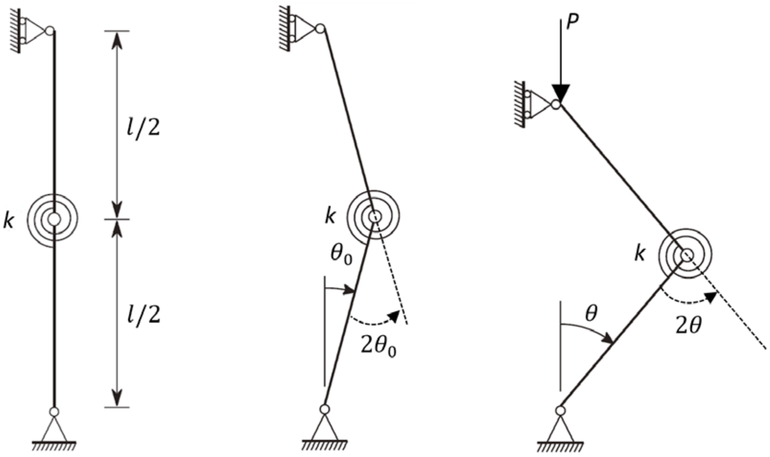


Figure 6 – Idealized system with a joint with a spring stiffness [2].

Assuming that the upper and lower bars have an initial rotation "θ_0", with zero rotation of the spring, and an axial load is applied, the rotation increases to θ, and the moment on the spring becomes $M_{spring} = k{\cdot}2(\theta - \theta_0)$, where k is the (elastic) spring stiffness. The equilibrium in the deformed shape leads to the following expression: $P\theta\, l/2 = M_{spring}$. From the two previous expressions, it can be shown that $P = \frac{4k}{l}\left(\frac{\theta - \theta_0}{\theta}\right)$. The critical buckling load P_{cr} is for a perfectly straight member, i.e. $\theta_0 = 0$. In this case, $P_{cr} = 4k/l$. Therefore, $P = P_{cr}\left(\frac{\theta - \theta_0}{\theta}\right)$. If $\theta_0 \neq 0$, θ would need to be infinite for P to be equal to P_{cr}. This means that the imperfect column will never reach the Euler load (this is consistent with the line OCGAB from Figure 4). The equation can be re-written as $\theta = \left(\frac{1}{1-\mu}\right)\theta_0$, where $\mu = P_{cr}/P$. This is the so-called amplification factor. This factor allows the consideration of second order effects by amplifying the first order effects. EN 1993-1-1 section 5.2.2 introduces this factor for frame stability in the form of $\frac{1}{1-1/\alpha_{cr}}$ which leads to $\alpha_{cr} = P_{cr}/P$, where P is the applied load and P_{cr} is the elastic critical load (for a strut, this will be Euler load). From a rigorous calculation, it can be justified that the simplified formulation provides reasonable results for $P \leq 0.5P_{cr}$ ($\alpha_{cr} \geq 2$)[7]. EN 1993-1-1 clause 5.2.2 limits the method for frame applications where $\alpha_{cr} \geq 3$.

The global *P-Δ* effects, according to clause 5.2.1 of EN 1993-1-1 need to be considered for the cases where the value of $\alpha_{cr} \leq 10$ for an elastic global analysis, and $\alpha_{cr} \leq 15$ for a plastic global analysis. Global imperfections for frames are defined according to EN 1993-1-1 section 5.3.2. Basically, an initial frame rotation $\phi = h/200$ (where h is the height of the frame/structure) is recommended (Figure 1), although the value can be reduced based on the number of columns and height of the frame. If the applied horizontal loads in the frame are more than 15% of the vertical loads, clause 5.3.2 of EN 1993-1-1 allows the global imperfections to be neglected. In this circumstance, the effects of global imperfections are small compared to that of the applied horizontal loads.

To assess global instability in a structure, the problem is often addressed using the Finite Element Method. In simple terms, the stiffness of a beam element is reduced based on the level of axial force. The method leads to a stiffness matrix $[K^t]$ for the total structure, where the critical factor α_{cr} is obtained by solving the determinant $|K^t| = 0$. Different buckling modes can be found (eigenvalues). For global stability, local modes (related to individual members) are ignored. The exact answer for the problem is complex, leading to the implementation of simplified approaches. The exact answer for a simple beam with no axial or shear deformation is presented in Figure 7. The terms in the matrix depend on the stability functions ϕ_i. By necessity, simplification generally involves making approximation to the highly non-linear ϕ_i functions (see Figure 8), which in turn leads to recommendations regarding modelling.

At large values of N/P_{cr}, the difference between precise and approximate values for ϕ_i is significant. It is therefore recommended that individual members are modelled by at least 3 finite elements, which reduces the N/P_{cr} ratio by a factor of 9, and consequently reduces the error in taking approximated values for ϕ_i. The maximum value of N/P_{cr} is 4 (when $l_{eff} = 0.5l$), so modelling the member with 3 finite elements reduces the ratio to 0.44. As can be seen from Figure 8, the error between the approximate and precise values of ϕ_i functions for $N/P_{cr} = 0.44$ is insignificant.

$$\left[K^t_{ij}\right] = \frac{EI}{l}\begin{bmatrix} 4\phi_3 & \frac{6\phi_2}{l} & 2\phi_4 & \frac{-6\phi_2}{l} \\ \frac{6\phi_2}{l} & \frac{12\phi_1}{l^2} & \frac{6\phi_2}{l} & \frac{-12\phi_1}{l^2} \\ 2\phi_4 & \frac{6\phi_2}{l} & 4\phi_3 & \frac{-6\phi_2}{l} \\ \frac{-6\phi_2}{l} & \frac{-12\phi_1}{l^2} & \frac{-6\phi_2}{l} & \frac{12\phi_1}{l^2} \end{bmatrix}$$

$$\phi_1 = \beta\phi_2 cotg(\beta)$$
$$\phi_2 = \frac{\beta^2}{3[1 - \beta cotg(\beta)]}$$
$$\phi_3 = \frac{3}{4}\phi_2 + \frac{1}{4}\beta cotg(\beta)$$
$$\phi_4 = \frac{3}{2}\phi_2 + \frac{1}{2}\beta cotg(\beta)$$
$$\beta = \frac{\pi}{2}\sqrt{\frac{N}{P_{cr}}}$$

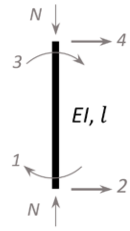


Figure 7 – Formulation for the exact stiffness matrix [8,9].

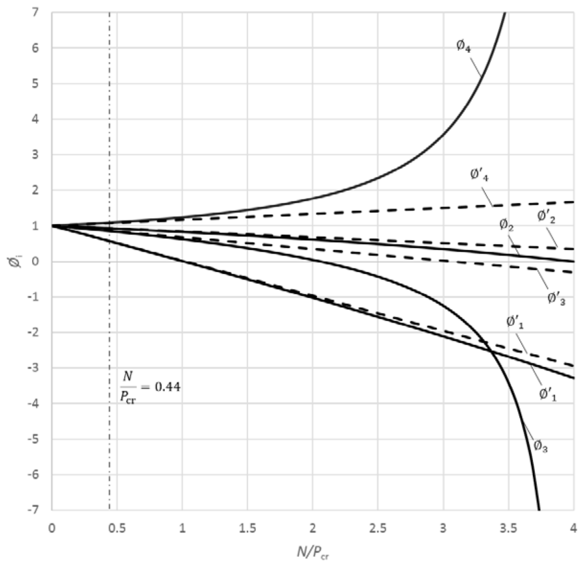


Figure 8 – Stability functions for exact (ϕ_i solid lines) and for approximate (ϕ'_i – dashed lines) stiffness matrix stiffness (ϕ'_i) [8,9].

Conclusions

1. Buckling problems demand the consideration of the deformed shape of the system;
2. The concept of an effective length is used to adapt the Euler buckling load to different boundary conditions;
3. An imperfect strut buckles before the plastic section capacity is reached;
4. Elastic section modulus must be used to back-calculate the initial imperfection;
5. Second order effects can be allowed for by using an amplification factor;
6. Approximate methods for stability functions ϕ_i are generally used in assessing frame stability;
7. Modelling with at least three finite elements per member reduces the error in using approximate stability functions.

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